LINEAR NESTED ARTIN APPROXIMATION THEOREM FOR ALGEBRAIC POWER SERIES

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Abstract. We give a new and elementary proof of the nested Artin approximation Theorem for linear equations with algebraic power series coefficients. Moreover, for any Noetherian local subring of the ring of formal power series, we clarify the relationship between this theorem and the problem of the commutation of two operations for ideals: the operation of replacing an ideal by its completion and the operation of replacing an ideal by one of its elimination ideals.

1. Introduction

The aim of the paper is to investigate the nested Artin approximation problem for linear equations. Namely the problem is the following: if

$$F(x, y) = 0$$

is a system of algebraic or analytic equations which are linear in y, with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, and if $y(x)$ is a formal power series solution

$$F(x, y(x)) = 0$$

with the property that

$$y_i(x) \text{ depends only on the variables } x_1, \ldots, x_{\sigma_i} \tag{1.1}$$

for some integers σ_i, is it possible to find algebraic or analytic solutions satisfying (1.1)?

In this paper we provide a characterization for a certain class of germs of functions to satisfy the nested Artin approximation property and we prove that the rings of algebraic power series satisfy this property.

In order to explain the situation let us consider the following theorem (proved by M. Artin in characteristic zero and by M. André in positive characteristic):

Theorem 2.1 [Ar68][An75] Let $\Bbbk$ be a complete valued field and let $F(x, y)$ be a vector of convergent power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x)$ vanishing at 0,

$$F(x, \widehat{y}(x)) = 0.$$

Then, for any $c \in \mathbb{N}$, there exists a convergent power series solution $\widetilde{y}(x)$,

$$F(x, \widetilde{y}(x)) = 0$$

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which coincides with $\widehat{y}(x)$ up to degree c,

$$\widetilde{y}(x) \equiv \widehat{y}(x) \text{ modulo } (x)^c.$$

Then M. Artin (see [Ar71, p.7]) asked, whether or not, given a formal solution $\widehat{y}(x) = (\widehat{y}_1(x), \dots, \widehat{y}_m(x))$ satisfying

$$\widehat{y}_j(x) \in \Bbbk[\![x_1, \dots, x_{\sigma_j}]\!] \quad \forall j$$

for some integers $\sigma_j \in \{1, \dots, n\}$, there exists a convergent solution $\widetilde{y}(x)$ as in Theorem 2.1 such that

$$\widetilde{y}_j(x) \in \Bbbk\{x_1, \dots, x_{\sigma_j}\} \;\; \forall j.$$

Shortly after, A. Gabrielov [Ga71] gave an example showing that the answer to the previous question is negative in general. In fact this example is built from a counterexample to a conjecture of A. Grothendieck he gave in [Ga71]. This conjecture of Grothendieck is the following (see [Gr60, p. 13-08]):

If $\varphi : \mathbb{C}\{x\}/I \longrightarrow \mathbb{C}\{y\}/J$ is an injective morphism of analytic algebras then the corresponding morphism $\varphi : \mathbb{C}[\![x]\!]/I\mathbb{C}[\![x]\!] \longrightarrow \mathbb{C}[\![y]\!]/J\mathbb{C}[\![y]\!]$ is again injective.

Even if it is obvious that the counterexample of Gabrielov to Grothendieck's conjecture provides a negative answer to the question of M. Artin, the relationship between these two problems is not clear in general.

The main goal of this note is to clarify that relationship between Grothendieck's conjecture and Artin's question. We show in a general frame (i.e. not only for the rings of convergent power series but for more general families of rings - cf. Definition 3.1) that Grothendieck's conjecture is equivalent to the question of M. Artin in the case where $F(x, y)$ is linear in y (see Theorem 3.9). Let us mention that it is well known that Grothendieck's conjecture is equivalent to Artin's question for some very particular $F(x, y)$ which are linear in y (see [Be77] and [Ro08]) but it was not known that they are equivalent for all $F(x, y)$ linear in y.

We also prove (see Theorem 3.9) that these two problems are equivalent to the problem of the commutation of two operations: the operation of replacing an ideal by its completion and the operation of replacing an ideal by one of its elimination ideals (see 3.2).

Moreover, we consider the particular case of the rings of algebraic power series. Let us recall that a formal power series $f(x) \in \Bbbk[\![x_1, \dots, x_n]\!]$ is called *algebraic* if it is algebraic over the ring of polynomials $\Bbbk[x_1, \dots, x_n]$. The ring of algebraic power series is denoted by $\Bbbk\langle x_1, \dots, x_n\rangle$. Indeed after A. Gabrielov gave a negative answer to both problems, D. Popescu showed that Artin's question has a positive answer in the case the ring of convergent power series is replaced by the ring of algebraic power series:

Theorem 2.2 [Po86] Let $\Bbbk$ be a field and $F(x, y)$ be a vector of algebraic power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x) = (\widehat{y}_1(x), \dots, \widehat{y}_m(x))$ vanishing at 0,

$$F(x, \widehat{y}(x)) = 0.$$

Moreover let us assume that $\widehat{y}_j \in \Bbbk[\![x_1, \dots, x_{\sigma_j}]\!]$, $1 \leq j \leq m$, for some integers σ_j, $1 \leq \sigma_j \leq n$.

Then for any $c \in \mathbb{N}$ there exists an algebraic power series solution $\widetilde{y}(x)$ such that for all j, $\widetilde{y}_j(x) \in \Bbbk\langle x_1, \dots, x_{\sigma_j}\rangle$ and $\widetilde{y}(x) - \widehat{y}(x) \in (x)^c$.

Let us remark that if $F(x,y)$ is a vector of polynomials in y with coefficients in $\Bbbk\langle x\rangle$ we may drop the condition that $\widehat{y}(x)$ vanishes at 0 by replacing $F(x,y)$ (resp. $\widehat{y}(x)$) by $F(x,y+\widehat{y}(0))$ (resp. $\widehat{y}(x)-\widehat{y}(0)$). This result has a large range of applications (see [FB12], [Mir12] or [Sh10] for some recent examples) but its proof is based on the so-called General Néron Desingularization which is quite involved.

In the second part of this paper we provide a new and elementary proof of Theorem 2.2 for equations $F(x,y)=0$ which are linear in y (see Theorem 3.11). This shows that Theorem 2.2 is really easier in the case $F(x,y)$ is linear in y.

Finally we mention that the question of Grothendieck has been widely studied in the case of convergent power series rings and it has been shown that the answer is positive for some particular cases (see for instance [AvdP70], [Ga73], [EH77], [Mil78], [Iz89] or [To90]). One of them, similar to our situation, is the case of a morphism $\varphi:\Bbbk\{x\}/I\longrightarrow\Bbbk\{y\}/J$ where the images of the x_i are algebraic power series and the ideals I and J are prime and generated by algebraic power series (it has been proven in several steps in [To76], [Be77], [Mil78] and [Ro09]). For this kind of morphisms it is shown that φ is injective if and only if $\widehat{\varphi}$ is injective.

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3. Definitions and main results

Definition 3.1. Let $\Bbbk$ be a field. An *admissible family of rings* is an increasing sequence of rings $\mathcal{F}=(R_n)_{n\in\mathbb{N}}$ satisfying the following properties:

(1) For every integer $n\geq 0$ the ring R_n is a $\Bbbk$-subalgebra of $\Bbbk[\![x_1,\ldots,x_n]\!]$ (in particular $R_0=\Bbbk$).
(2) For every integer $n\geq 0$, $\Bbbk[x_1,\ldots,x_n]\subset R_n$.
(3) For every integer $n>0$ the ring R_n is a Noetherian local ring whose maximal ideal is generated by $x_1,\ldots,x_n$.
(4) For every integers m,n with $0\leq m\leq n$ we have
$$R_n\cap\Bbbk[\![x_1,\ldots,x_m]\!]=R_m.$$

When an admissible family of rings is given, any element of a member of this family is called an *admissible power series*.

Sometimes we will emphasize the dependency of R_n on the variables $(x_1,\ldots,x_n)$ by writing $R_n=\Bbbk\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle$ for $n\in\mathbb{N}$.

Example 3.2. The following families of rings are admissible:

- The rings of convergent power series over a valued field $\Bbbk$.
- The rings of algebraic power series over a field $\Bbbk$.

- The rings of formal power series.
- The rings of germs of rational functions at $0 \in \Bbbk^n$, $\Bbbk[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$.

3.1. **Krull topology.** Let $(A, \mathfrak{m})$ be a Noetherian local ring. The *Krull topology* of A is the topology in which the ideals $\mathfrak{m}^c$ constitute a basis of neighborhoods of the zero of A. For a A-module M the Krull topology of M is the one in which the submodules $\mathfrak{m}^c M$ constitute a basis of neighborhoods of the zero of M. The completion of A (resp. M) for the Krull topology is denoted by $\widehat{A}$ (resp. $\widehat{M}$). We have the following lemma asserting that the topological closure of a finite module and its completion coincide:

Lemma 3.3. *(*[SZ58, Corollary 2, p. 257]*) If N is a A-submodule of a finite A-module M then the closure of N in $\widehat{M}$ is $\widehat{N} = \widehat{A}N$.*

Definition 3.4. If M is a A-module where $(A, \mathfrak{m})$ is a Noetherian local ring and E is a subset of M, we say that an element $f \in M$ may *be approximated by elements of E* if f is in the closure (for the Krull topology) of E in M, i.e. if for every integer c there exists $f_c \in E$ such that $f - f_c \in \mathfrak{m}^c M$.

3.2. **Strong elimination property.** One says that an admissible family of rings $\mathcal{F} = (\Bbbk\langle\langle x_1, \dots, x_n\rangle\rangle)_n$ has the *strong elimination property* if for any two sets of variables x and y and any ideal I of $\Bbbk\langle\langle x, y\rangle\rangle$ we have

$$(I \cap \Bbbk\langle\langle x\rangle\rangle)\Bbbk[\![x]\!] = \widehat{I} \cap \Bbbk[\![x]\!] \tag{3.1}$$

where $\widehat{I}$ denotes the ideal of $\Bbbk[\![x, y]\!]$ generated by I.

Remark 3.5. Since $I \cap \Bbbk\langle\langle x\rangle\rangle \subset \widehat{I} \cap \Bbbk[\![x]\!]$, Lemma 3.3 shows that (3.1) is equivalent to say that the elements of $\widehat{I} \cap \Bbbk[\![x]\!]$ may be approximated by elements of $I \cap \Bbbk\langle\langle x\rangle\rangle$.

3.3. **Linear nested approximation property.** We say that an admissible family of rings $\mathcal{F} = (\Bbbk\langle\langle x_1, \dots, x_n\rangle\rangle)_n$ has the *linear nested approximation property* if the following property holds:
Let m, n, p be positive integers, T be a $p \times m$ matrix with entries in $\Bbbk\langle\langle x\rangle\rangle := \Bbbk\langle\langle x_1, \dots, x_n\rangle\rangle$, $b = (b_1, \dots, b_p) \in \Bbbk\langle\langle x\rangle\rangle^p$ and $\sigma : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$ be a weakly increasing function. Let $y = (y_1, \dots, y_m)$ be a vector of new variables. Then the set of solutions $y(x)$ in

$$\Bbbk\langle\langle x_1, \dots, x_{\sigma(1)}\rangle\rangle \times \cdots \times \Bbbk\langle\langle x_1, \dots, x_{\sigma(m)}\rangle\rangle$$

of the following system of linear equations

$$Ty = b \tag{S}$$

is dense in the set of formal solutions in

$$\Bbbk[\![x_1, \dots, x_{\sigma(1)}]\!] \times \cdots \times \Bbbk[\![x_1, \dots, x_{\sigma(m)}]\!].$$

3.4. **Strongly injective morphisms.**

Definition 3.6. Let $\varphi : A \longrightarrow B$ be a morphism of local rings. We denote by $\widehat{\varphi}$ the induced morphism $\widehat{A} \longrightarrow \widehat{B}$. One says that φ is *strongly injective* if $\widehat{\varphi}$ is injective.

Definition 3.7. We say that an admissible family of rings $\mathcal{F} = (\Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle)_n$ has the *strong injectivity property* if for any integers n and m and any ideals I of $\Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle$ and J of $\Bbbk\langle\langle y_1, \ldots, y_m\rangle\rangle$, any injective morphism

$$\frac{\Bbbk\langle\langle x\rangle\rangle}{I} \longrightarrow \frac{\Bbbk\langle\langle y\rangle\rangle}{J}$$

is strongly injective.

Remark 3.8. Definition 3.6 is not the classical one. In [AvdP70] a morphism $\varphi : A \longrightarrow B$ is called strongly injective if $\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A)$. This definition, which is the classical one, is stronger than the one we use in this paper. Nevertheless we will prove that if an admissible family of rings $(\Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle)_n$ has the strong injectivity property then for any morphism $\varphi : A = \frac{\Bbbk\langle\langle x\rangle\rangle}{I} \longrightarrow B = \frac{\Bbbk\langle\langle y\rangle\rangle}{J}$ we have $\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A)$ (see Corollary 3.10).

3.5. **Main results.** The first main result of this paper is the following:

Theorem 3.9. *For an admissible family of rings $\mathcal{F} = (\Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle)_n$ the following properties are equivalent:*

(i) *$\mathcal{F}$ has the strong elimination property.*
(ii) *$\mathcal{F}$ has the linear nested approximation property.*
(iii) *$\mathcal{F}$ has the strong injectivity property.*

Corollary 3.10. *Let $\mathcal{F} = (\Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle)_n$ be an admissible family having the strong injectivity property. Then for any morphism*

$$\varphi : A = \frac{\Bbbk\langle\langle x\rangle\rangle}{I} \longrightarrow B = \frac{\Bbbk\langle\langle y\rangle\rangle}{J}$$

we have

$$\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A).$$

Proof. Clearly $\varphi(A) \subset \widehat{\varphi}(\widehat{A}) \cap B$. Let us prove the reverse inclusion.
Let $\widehat{f} \in \widehat{A}$ such that $\widehat{\varphi}(\widehat{f}) = b \in B$. Let us denote by $\varphi_i(y)$ an admissible power series of $\Bbbk\langle\langle y\rangle\rangle$ which is the image of x_i by φ modulo J, for $i = 1, \ldots, n$. Let $p_1(x), \ldots, p_r(x)$ be generators of I and $q_1(y), \ldots, q_s(y)$ be generators of J. Thus, by assumption, there exist formal power series

$$\widehat{h}_\ell, \widehat{l}_k, \widehat{k}_i$$

such that

$$\widehat{f}(x) + \sum_{\ell=1}^{r} p_\ell(x)\widehat{h}_\ell(x, y) = b(y) + \sum_{k=1}^{s} q_k(y)\widehat{l}_k(x, y) + \sum_{i=1}^{n}(x_i - \varphi_i(y))\widehat{k}_i(x, y).$$

By the previous theorem the family of rings $\mathcal{F}$ has the linear nested approximation property, thus there exist admissible power series

$$f(x), h_\ell(x, y), l_k(x, y), k_i(x, y)$$

such that

$$f(x) + \sum_{\ell=1}^{r} p_\ell(x)h_\ell(x, y) = b(y) + \sum_{k=1}^{s} q_k(y)l_k(x, y) + \sum_{i=1}^{n}(x_i - \varphi_i(y))k_i(x, y).$$

In particular, by replacing x_i by $\varphi_i(y)$ for all i we see that

$$\varphi(f) = b.$$

Thus $b \in \varphi(A)$. □

Our second main result is the following:

Theorem 3.11. *The family $(\Bbbk\langle x_1, \ldots, x_n\rangle)_n$ of algebraic power series rings over a field $\Bbbk$ satisfies the equivalent properties of Theorem 3.9.*

Remark 3.12. Let $\mathcal{F} = (R_n)_n$ be an admissible family. Let $f \in R_n$ such that $f(0) = 0$ and $\frac{\partial f}{\partial x_n}(0) \neq 0$. By the implicit function Theorem for formal power series there exists a unique formal power series $h(x')$ with $x' = (x_1, \ldots, x_{n-1})$ such that

$$f(x', h(x')) = 0 \text{ and } h(0) = 0.$$

Thus, by Taylor's formula, there exists a formal power series $g(x)$ such that

$$f(x) + (x_n - h(x'))g(x) = 0.$$

Since $\frac{\partial f}{\partial x_n}(0) \neq 0$ and $h(0) = 0$ we have $g(0) \neq 0$, i.e. $g(x)$ is a unit. Hence we have, where $u(x)$ denotes the inverse of $g(x)$:

$$f(x)u(x) + x_n - h(x') = 0.$$

Moreover, since $h(x')$ is unique, $u(x)$ is also unique and the linear equation

$$f(x)y_2 + x_n - y_1 = 0$$

has a unique nested formal solution $(h(x'), u(x))$ whose first component vanishes at 0. Thus if the family $\mathcal{F}$ satisfies the equivalent properties of Theorem 3.9 then this family has to satisfy the implicit function Theorem (which is equivalent to say that the rings R_n are Henselian local rings).
In particular the family of germs of rational functions at the origin of $\Bbbk^n$ does not satisfy the properties of Theorem 3.9.
Since the ring of algebraic power series in n variables is the Henselization of the ring of germs of rational functions at the origin of $\Bbbk^n$, this also shows that the family of algebraic power series is the smallest admissible family containing the family of germs of rational functions at the origin of $\Bbbk^n$ and satisfying the properties of Theorem 3.9.

Remark 3.13. Let $\mathcal{F} = (R_n)_n$ be an admissible family and f, g two elements of R_n. Let us assume that f is x_n-regular of order d, i.e. $f(0, x_n) = x_n^d u(x_n)$ for some unit $u(x_n)$. By the Weierstrass division Theorem for formal power series there exists a unique vector

$$(q(x), a_0(x'), \ldots, a_{d-1}(x')) \in \Bbbk[\![x]\!] \times \Bbbk[\![x']\!]^d$$

with $x' = (x_1, \ldots, x_{n-1})$ such that

$$g(x) = f(x)q(x) + \sum_{\kappa=0}^{d-1} a_\kappa(x')x_n^\kappa.$$

By the uniqueness of $(q(x), a_0(x'), \ldots, a_{d-1}(x'))$ if the family $\mathcal{F}$ has the linear nested approximation property then

$$(q(x), a_0(x'), \ldots, a_{d-1}(x')) \in R_n \times R_{n-1}^d.$$

Thus $\mathcal{F}$ satisfies the Weierstrass division Theorem if it satisfies the equivalent properties of Theorem 3.9.

Remark 3.14. The example of Gabrielov [Ga71] shows that the family of convergent power series over a characteristic zero valued field does not satisfy the properties of Theorem 3.9 (but this family satisfies the implicit function Theorem, it even satisfies the Weierstrass division Theorem). This example is the following one: Let

$$\varphi : \mathbb{C}\{x_1, x_2, x_3\} \longrightarrow \mathbb{C}\{y_1, y_2\}$$

be the morphism of analytic $\mathbb{C}$-algebras defined by

$$\varphi(x_1) = y_1,\ \varphi(x_2) = y_1 y_2,\ \varphi(x_3) = y_1 e^{y_2}.$$

Then it is not very difficult to show that φ and $\widehat{\varphi}$ are both injective (see [Os16]). Then A. Gabrielov remarked that there exists a formal but not convergent power series $\widehat{g}(x)$ whose image $h(y)$ by $\widehat{\varphi}$ is convergent (see [Ga71]). This shows that Corollary 3.10 is not satisfied for convergent power series rings. Thus the properties of Theorem 3.9 are not satisfied in the case of convergent power series rings.

4. Proof of Theorem 3.9

We will prove the following implications:

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(i)}$$

The main difficulty is the first implication.

4.1. **Proof of (i) $\Longrightarrow$ (ii).** We assume that $\mathcal{F}$ has the strong elimination property and we fix a system of linear equations as (S). We call an *admissible nested solution* (resp. *formal nested solution*) of such a system (S) a solution in

$$\Bbbk\langle\!\langle x_1, \ldots, x_{\sigma(1)}\rangle\!\rangle \times \cdots \times \Bbbk\langle\!\langle x_1, \ldots, x_{\sigma(m)}\rangle\!\rangle$$

$$(\text{resp. } \Bbbk[\![x_1, \ldots, x_{\sigma(1)}]\!] \times \cdots \times \Bbbk[\![x_1, \ldots, x_{\sigma(m)}]\!]).$$

We will show that the set of admissible nested solutions is dense, for the $\mathfrak{m}$-adic topology, in the set of formal nested solutions.

Remark 4.1. If $\sigma(m) < n$ the coefficients of the powers of x_n in the equations (S) yield a new set of linear equations with coefficients in $x_1, \ldots, x_{n-1}$. By Noetherianity this set of equations is finitely generated. Thus by induction we may assume that

$$\sigma(m) = n. \tag{4.1}$$

• First we claim that we can assume that $b = 0$, i.e. the system (S) of linear equations is homogeneous. Indeed let us assume that the set of admissible nested solutions of any linear homogeneous system is dense in the set of formal nested solutions and let us fix a linear (non-homogenous) system as (S). Let $y(x) \in \Bbbk[\![x]\!]^m$ be a formal nested solution of the system (S): $Ty = b$.
Let us write $a_{i,j}$ the entries of the $p \times m$ matrix T and denote by T' the matrix

$$T' = \left[\ -b\,|\,T\ \right]$$

and set $y' = (y_0, y_1, \ldots, y_m)$.

Let us extend the previous function σ to $\{0,\dots,m\}$ by $\sigma(0)=\sigma(1)$. Since $y(x)$ is a formal nested solution of (S), $y'(x)=(1,y(x))$ is a formal nested solution of the following linear homogeneous system:

$$T'y'=0 \tag{S'}$$

By assumption, for any given integer $c\geq 1$, there exists an admissible nested solution $y'_c(x)=(y_{0,c}(x),y_{1,c}(x),\dots,y_{m,c}(x))$ of (S') such that

$$y_{0,c}(x)-1\in (x)^c \text{ and } y_{j,c}(x)-y_j(x)\in(x)^c \quad \forall j\geq 1.$$

In particular $y_{0,c}(0)=1\neq 0$ and $y_{0,c}(x)$ is a unit. Thus

$$(y_{0,c}(x)^{-1}y_{1,c}(x),\dots,y_{0,c}(x)^{-1}y_{m,c}(x))$$

is an admissible nested solution of (S). Moreover, for all $j\geq 1$, we have:

$$y_{0,c}(x)^{-1}y_{j,c}(x)-y_j(x)=(y_{0,c}(x)^{-1}-1)y_{j,c}(x)+(y_{j,c}(x)-y_j(x))\in(x)^c.$$

Thus the set of admissible nested solutions of (S) is dense in the set of formal nested solutions of (S) and the claim is proven.

• Let us consider a homogeneous linear system (S) where $b=0$. The set of (non-nested) admissible solutions of such a system is a sub-$\Bbbk\langle\langle x\rangle\rangle$-module of $\Bbbk\langle\langle x\rangle\rangle^m$ denoted by M. By Noetherianity this module is finitely generated. The set of (non-nested) formal solutions is the completion of M denoted by $\widehat{M}$ (by flatness of $\Bbbk\langle\langle x\rangle\rangle\longrightarrow\Bbbk[\![x]\!]$ since $\Bbbk\langle\langle x\rangle\rangle$ is a Noetherian local ring). Thus we are reduced to prove the following lemma:

Lemma 4.2. *Let M be a finite sub-module of $\Bbbk\langle\langle x\rangle\rangle^m$. Then*

$$M\cap\left(\Bbbk\langle\langle x_1,\dots,x_{\sigma(1)}\rangle\rangle\times\dots\times\Bbbk\langle\langle x_1,\dots,x_{\sigma(m)}\rangle\rangle\right)$$

is dense in

$$\widehat{M}\cap\left(\Bbbk[\![x_1,\dots,x_{\sigma(1)}]\!]\times\dots\times\Bbbk[\![x_1,\dots,x_{\sigma(m)}]\!]\right).$$

In order to prove Lemma 4.2 we may assume that $\sigma(m)=n$ by (4.1). We will prove Lemma 4.2 by induction on n.

Proof of Lemma 4.2. We take a module M as in Lemma 4.2 and we assume that $\sigma(m)=n$. Let us assume that Lemma 4.2 is proven for any submodule of the free module $\Bbbk\langle\langle x_1,\dots,x_{n'}\rangle\rangle^{m'}$ for any integers $n'<n$ and $m'\geq 1$.

Let $e_1,\dots,e_m$ be the canonical basis of $\Bbbk\langle\langle x\rangle\rangle^m$. Let $r\leq m-1$ be the integer such that for $j\in\{1,\dots,m\}$

$$\sigma(j)<n \ \ \forall j\leq r,\quad \sigma(j)=n\ \ \forall j>r.$$

We assume that M is generated by some vectors $\omega_1,\dots,\omega_s$ in $\Bbbk\langle\langle x\rangle\rangle^m$.
So Lemma 4.2 is equivalent to say that for any formal nested solution

$$\left(v_1(x),\dots,v_s(x),y_1(x_1,\dots,x_{\sigma(1)}),\dots,y_m(x_1,\dots,x_{\sigma(m)})\right)$$

of

$$\sum_{k=1}^{s}v_k(x)\omega_k=\sum_{j=1}^{m}y_j(x_1,\dots,x_{\sigma(j)})e_j \tag{4.2}$$

and any integer c there exists a formal nested solution of (4.2)

$$\big(v_{1,c}(x),\dots,v_{s,c}(x),y_{1,c}(x_1,\dots,x_{\sigma(1)}),\dots,y_{m,c}(x_1,\dots,x_{\sigma(m)})\big)$$

such that

- $y_{j,c}(x_1,\dots,x_{\sigma(j)})\in\Bbbk\langle\langle x_1,\dots,x_{\sigma(j)}\rangle\rangle$ for $j=1,\dots,m$
- $\displaystyle\sum_{k=1}^{s}v_{k,c}(x)\omega_k\in M$
- The series $v_{k,c}(x)-v_k(x)$, $y_{j,c}(x_1,\dots,x_{\sigma(j)})-y_j(x_1,\dots,x_{\sigma(j)})$ belong to the ideal $(x)^c$ for all k and j.

The morphism $\Bbbk\langle\langle x\rangle\rangle\longrightarrow\Bbbk[\![x]\!]$ being faithfully flat we may even assume that $v_{k,c}(x)\in\Bbbk\langle\langle x\rangle\rangle$ for all k. Thus Lemma 4.2 is equivalent to say that the set of admissible nested solutions of (4.2) is dense in the set of formal nested solutions of (4.2).

Since (4.2) is equivalent to

$$\sum_{k=1}^{s}v_k(x)\omega_k-\sum_{j=r+1}^{m}y_j(x)e_j=\sum_{j=1}^{r}y_i(x_1,\dots,x_{\sigma(j)})e_j \tag{4.3}$$

Lemma 4.2 is equivalent to the following statement (if we replace M by the $\Bbbk\langle\langle x\rangle\rangle$-module generated by M and $e_{r+1},\dots,e_m$):

Claim:

$$M\cap\big(\Bbbk\langle\langle x_1,\dots,x_{\sigma(1)}\rangle\rangle\cdot e_1+\dots+\Bbbk\langle\langle x_1,\dots,x_{\sigma(r)}\rangle\rangle\cdot e_r\big)$$

is dense in

$$\widehat{M}\cap\big(\Bbbk[\![x_1,\dots,x_{\sigma(1)}]\!]\cdot e_1+\dots+\Bbbk[\![x_1,\dots,x_{\sigma(r)}]\!]\cdot e_r\big).$$

Let us prove this claim. We assume that M is generated by the ω_k, $1\leq k\leq s$, where

$$\omega_k=(\omega_{k,1},\dots,\omega_{k,m})\in\Bbbk\langle\langle x\rangle\rangle^m.$$

Let S be the ring whose elements are the couples $(a,\omega)\in\Bbbk\langle\langle x\rangle\rangle\times\Bbbk\langle\langle x\rangle\rangle^m$ with addition and multiplication defined as follows:

$$(a,\omega)+(a',\omega')=(a+a',\omega+\omega'),$$

$$(a,\omega)(a',\omega')=(aa',a\omega'+a'\omega).$$

Then the ring S is isomorphic to

$$\frac{\Bbbk\langle\langle x,y_1,\dots,y_m\rangle\rangle}{(y)^2}$$

by the isomorphism defined by

$$(a,(a_1,\dots,a_m))\longmapsto a+a_1y_1+\dots+a_my_m.$$

This is the idealization principle of Nagata.

Let I be the ideal $\{0\}\times M$, i.e. the ideal of S generated by the elements

$$\omega_{k,1}y_1+\dots+\omega_{k,m}y_m\text{ for }k=1,\dots,s.$$

Let us define the ring (we set $x' := (x_1, \ldots, x_{\sigma(r)})$ and $y' := (y_1, \ldots, y_r)$)

$$R = \frac{\mathbb{k}\langle\langle x', y'\rangle\rangle}{(y')^2}.$$

Thus the previous claim is equivalent to the following assertion:

$$(I \cap R) \cap \left(\mathbb{k}\langle\langle x_1, \ldots x_{\sigma(1)}\rangle\rangle \cdot y_1 + \cdots + \mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(r)}\rangle\rangle \cdot y_r\right)$$

is dense in

$$(\widehat{I} \cap \widehat{R}) \cap \left(\mathbb{k}[\![x_1, \ldots x_{\sigma(1)}]\!] \cdot y_1 + \cdots + \mathbb{k}[\![x_1, \ldots, x_{\sigma(r)}]\!] \cdot y_r\right).$$

We can replace I by $I + (y)^2$, R by $\mathbb{k}\langle\langle x', y'\rangle\rangle$ and S by $\mathbb{k}\langle\langle x, y\rangle\rangle$. Then the previous statement is equivalent to:

$$(I \cap \mathbb{k}\langle\langle x', y'\rangle\rangle) \cap \left(\mathbb{k}\langle\langle x_1, \ldots x_{\sigma(1)}\rangle\rangle \cdot y_1 + \cdots + \mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(r)}\rangle\rangle \cdot y_r + \mathbb{k}\langle\langle x', y'\rangle\rangle \cdot (y')^2\right)$$

is dense in

$$(\widehat{I} \cap \mathbb{k}[\![x', y']\!]) \cap \left(\mathbb{k}[\![x_1, \ldots x_{\sigma(1)}]\!] \cdot y_1 + \cdots + \mathbb{k}[\![x_1, \ldots, x_{\sigma(r)}]\!] \cdot y_r + \mathbb{k}[\![x', y']\!] \cdot (y')^2\right).$$

Let J denote the ideal $I \cap \mathbb{k}\langle\langle x', y'\rangle\rangle$. By the strong elimination property that we have assumed to be satisfied we have

$$\widehat{J} = \widehat{I} \cap \mathbb{k}[\![x', y']\!].$$

Thus the previous density statement is equivalent to the following one:

$$J \cap \left(\mathbb{k}\langle\langle x_1, \ldots x_{\sigma(1)}\rangle\rangle \cdot y_1 + \cdots + \mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(r)}\rangle\rangle \cdot y_r + \mathbb{k}\langle\langle x', y'\rangle\rangle \cdot (y')^2\right)$$

is dense in

$$\widehat{J} \cap \left(\mathbb{k}[\![x_1, \ldots x_{\sigma(1)}]\!] \cdot y_1 + \cdots + \mathbb{k}[\![x_1, \ldots, x_{\sigma(r)}]\!] \cdot y_r + \mathbb{k}[\![x', y']\!] \cdot (y')^2\right).$$

Let

$$N \subset \mathbb{k}\langle\langle x'\rangle\rangle \cdot y_1 + \cdots + \mathbb{k}\langle\langle x'\rangle\rangle \cdot y_r$$

be the $\mathbb{k}\langle\langle x'\rangle\rangle$-module generated by the linear combinations of $y_1, \ldots, y_r$ which are in $J + (y')^2$. Then we see that Lemma 4.2 for the module M is equivalent to the following statement:

$$N \cap \left(\mathbb{k}\langle\langle x_1, \ldots x_{\sigma(1)}\rangle\rangle \cdot y_1 + \cdots + \mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(r)}\rangle\rangle \cdot y_r\right)$$

is dense in

$$\widehat{N} \cap \left(\mathbb{k}[\![x_1, \ldots x_{\sigma(1)}]\!] \cdot y_1 + \cdots + \mathbb{k}[\![x_1, \ldots, x_{\sigma(r)}]\!] \cdot y_r\right).$$

But this statement is exactly Lemma 4.2 where n is replaced by $\sigma(r) < n$. Thus by the induction hypothesis the claim is proven. Hence Lemma 4.2 is true (since it is obviously true when $\sigma(m) = n = 0$). $\square$

4.2. **Proof of (ii) $\Longrightarrow$ (iii).** Let

$$\varphi : \frac{\Bbbk\langle\!\langle x\rangle\!\rangle}{I} \longrightarrow \frac{\Bbbk\langle\!\langle y\rangle\!\rangle}{J}$$

be an injective morphism and let $\widehat{f} \in \mathrm{Ker}(\widehat{\varphi})$. The morphism φ is defined by admissible power series $\varphi_1(y), \ldots, \varphi_n(y)$ such that

$$g(\varphi_1(y), \ldots, \varphi_n(y)) \in J \quad \forall g \in I$$

and, for any power series g, the image of g modulo I is equal to

$$g(\varphi_1(y), \ldots, \varphi_n(y)) \text{ modulo } J.$$

We still denote by $\widehat{f}$ a lifting of $\widehat{f}$ in $\Bbbk[\![x]\!]$. Thus

$$\widehat{f}(\varphi_1(y), \ldots, \varphi_n(y)) \in \widehat{J},$$

i.e. there exist formal power series $\widehat{h}_1(y), \ldots, \widehat{h}_s(y)$ such that

$$\widehat{f}(\varphi_1(y), \ldots, \varphi_n(y)) = \sum_{k=1}^{s} q_k(y)\widehat{h}_k(y)$$

where the $q_k(y)$ are generators of the ideal J. By Taylor's formula there exist formal power series $\widehat{k}_i(x, y)$ such that

$$\widehat{f}(x) - \sum_{k=1}^{s} q_k(y)\widehat{h}_k(y) = \sum_{i=1}^{n} (x_i - \varphi_i(y))\widehat{k}_i(x, y). \tag{4.4}$$

By the linear nested approximation property, for any integer c, there exists a vector of admissible power series

$$(f_c(x), h_{1,c}(x, y), \ldots, h_{s,c}(x, y), k_{1,c}(x, y), \ldots, k_{n,c}(x, y))$$

such that

$$f_c(x) - \sum_{k=1}^{s} q_k(y)h_{k,c}(x, y) = \sum_{i=1}^{n} (x_i - \varphi_i(y))k_{i,c}(x, y)$$

and

$$f_c(x) - \widehat{f}(x) \in (x)^c,\ \ h_{k,c}(x, y) - \widehat{h}_k(y) \in (x, y)^c,\ \ k_{i,c}(x, y) - \widehat{k}_i(x, y) \in (x, y)^c$$

for all k and i. By replacing x_i by $\varphi_i(y)$ for $i = 1, \ldots, n$, we see that $\varphi(f_c(x)) = 0$, thus $f_c(x) = 0$ since φ is injective. Thus $\widehat{f}(x) \in (x)^c$ for all $c \geq 0$ thus $\widehat{f}(x) = 0$ by Nakayama's Lemma. This shows that φ is strongly injective.

4.3. **Proof of (iii) $\Longrightarrow$ (i).** Let I be an ideal of $\Bbbk\langle\!\langle x, y\rangle\!\rangle$. Let φ be the following injective morphism induced by the inclusion $\Bbbk\langle\!\langle x\rangle\!\rangle \longrightarrow \Bbbk\langle\!\langle x, y\rangle\!\rangle$:

$$\frac{\Bbbk\langle\!\langle x\rangle\!\rangle}{I \cap \Bbbk\langle\!\langle x\rangle\!\rangle} \longrightarrow \frac{\Bbbk\langle\!\langle x, y\rangle\!\rangle}{I}.$$

Then $(I \cap \Bbbk\langle\!\langle x\rangle\!\rangle)\Bbbk[\![x]\!] = \widehat{I} \cap \Bbbk[\![x]\!]$ if and only if φ is strongly injective since

$$\mathrm{Ker}(\widehat{\varphi}) = \frac{\widehat{I} \cap \Bbbk[\![x]\!]}{(I \cap \Bbbk\langle\!\langle x\rangle\!\rangle)\Bbbk[\![x]\!]}.$$

5. Proof of Theorem 3.11

We will prove that the family of algebraic power series rings satisfies the strong elimination property. We set $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ and let I be an ideal of $\Bbbk\langle x, y\rangle$. Let $f_1(x, y), \ldots, f_r(x, y) \in \Bbbk\langle x, y\rangle$ be a set of generators of I. We need to prove that

$$(I \cap \Bbbk\langle x\rangle)\Bbbk[\![x]\!] = \widehat{I} \cap \Bbbk[\![x]\!].$$

By Remark 3.5 this is equivalent to say that any element $\widehat{u}(x)$ of $\widehat{I} \cap \Bbbk[\![x]\!]$ may be approximated by elements of $I \cap \Bbbk\langle x\rangle$. Such an element $\widehat{u}(x)$ has the form

$$\widehat{u}(x) = \sum_{j=1}^{r} f_j(x, y)\widehat{v}_j(x, y) \tag{5.1}$$

for some formal power series $\widehat{v}_j(x, y)$. By Proposition 5.1 given below, for any integer c there exist algebraic power series $u(x)$, $v_1(x, y), \ldots, v_r(x, y)$ such that

$$u(x) = \sum_{j=1}^{r} f_j(x, y)v_j(x, y)$$

and

$$u(x) - \widehat{u}(x) \in (x)^c, \;\; v_j(x, y) - \widehat{v}_j(x, y) \in (x, y)^c \quad \forall j.$$

This proves Theorem 3.11.

The next proposition has been proven by E. Bierstone and P. Milman in [BM87]. The proof we give here is a bit different but uses the same arguments as those of [BM87].

Proposition 5.1. *We write variables $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$, $u = (u_1, \ldots, u_q)$ and $v = (v_1, \ldots, v_r)$. For any $p \times (q + r)$ matrix T with entries in $\Bbbk\langle x, y\rangle$ and any vector $b \in \Bbbk\langle x, y\rangle^p$, every formal nested solution*

$$(\widehat{u}(x), \widehat{v}(x, y)) \in \Bbbk[\![x]\!]^q \times \Bbbk[\![x, y]\!]^r$$

of the linear system

$$T \begin{pmatrix} u \\ v \end{pmatrix} = b$$

may be approximated by algebraic nested solutions in $\Bbbk\langle x\rangle^q \times \Bbbk\langle x, y\rangle^r$.

Proof. Let

$$(\widehat{u}(x), \widehat{v}(x, y))$$

be a given nested formal solution of the linear system in the statement. Then $\widehat{v}(x, y)$ is a formal solution of the system

$$T'v = b' \tag{5.2}$$

where

$$T' = \begin{pmatrix} a_{1,q+1} & \cdots & a_{1,q+r} \\ a_{2,q+1} & \cdots & a_{2,q+r} \\ \vdots & \ddots & \vdots \\ a_{p,q+1} & \cdots & a_{p,q+r} \end{pmatrix},$$

$a_{i,j} := a_{i,j}(x,y) \in \Bbbk\langle x,y\rangle$ are the entries of T and

$$b'_i(x,y) := b_i(x,y) - \sum_{\kappa=1}^{q} a_{i,\kappa}(x,y)\widehat{u}_\kappa(x) \in \Bbbk[\![x,y]\!] \text{ for } i=1,\ldots,p.$$

The morphism $\Bbbk[\![x]\!]\langle y\rangle \longrightarrow \Bbbk[\![x,y]\!]$ being faithfully flat, for any integer c there exists a solution $\widetilde{v}(x,y) \in \Bbbk[\![x]\!]\langle y\rangle^r$ of (5.2) such that

$$\widetilde{v}(x,y) - \widehat{v}(x,y) \in (x,y)^c \Bbbk[\![x,y]\!]^r.$$

Thus from now on we may assume that $\widehat{v}(x,y) \in \Bbbk[\![x]\!]\langle y\rangle^r$. Moreover by doing the following linear change of coordinates

$$u_\kappa \longmapsto u_\kappa + \widehat{u}_\kappa(0),$$

we may assume that $\widehat{u}_\kappa(x) \in (x)\Bbbk[\![x]\!]$ for $\kappa = 1,\ldots,q$.
By Lemma 5.2 given below there exist a new set of variables $z = (z_1,\ldots,z_s)$, algebraic power series $g_\ell(y,z) \in \Bbbk\langle y,z\rangle$ for $1 \leq \ell \leq r$ and formal power series $\widehat{z}_1(x),\ldots,\widehat{z}_s(x) \in (x)\Bbbk[\![x]\!]$ such that

$$\widehat{v}_\ell(x,y) = g_\ell(y,\widehat{z}_1(x),\ldots,\widehat{z}_s(x)).$$

Thus, by replacing v_ℓ by $g_\ell(y,z)$ for $\ell = 1,\ldots,r$ in the system of equations $T\begin{pmatrix} u \\ v \end{pmatrix} = b$, we obtain a new system of (non linear) equations

$$f(x,y,\widehat{u}(x),\widehat{z}(x)) = 0$$

where $f(x,y,u,z)$ is a vector of algebraic power series.
Let $\mathcal{I}$ denote the ideal of $\Bbbk\langle x,u,z\rangle$ generated by all the coefficients of the monomials in y in the expansion of the components of f as power series in $(y_1,\ldots,y_m)$. Let $h_1,\ldots,h_t$ be a system of generators of $\mathcal{I}$. By assumption $(\widehat{u}(x),\widehat{z}(x))$ is a formal power series solution of the system

$$h_1(x,u,z) = \cdots = h_t(x,u,z) = 0. \tag{5.3}$$

Thus by Artin Approximation Theorem for algebraic power series [Ar69], for any integer $c \geq 0$ there exists $(\widetilde{u}(x),\widetilde{z}(x)) \in \Bbbk\langle x\rangle^{q+r}$ solution of the system (5.3) with

$$\widetilde{u}_\kappa(x) - \widehat{u}_\kappa(x) \in (x)^c, \widetilde{z}_k(x) - \widehat{z}_k(x) \in (x)^c \quad \forall \kappa, k.$$

Thus, by Taylor's formula,

$$\widetilde{v}_\ell(x,y) - \widehat{v}_\ell(x,y) \in (x,y)^c \quad \text{for } 1 \leq \ell \leq r$$

where

$$\widetilde{v}_\ell(x,y) = g_\ell(y,\widetilde{z}_1(x),\ldots,\widetilde{z}_k(x)) \quad \forall \ell.$$

Lemma 5.2. *(*[Ro15, Lemma 9.2]*) We have*

$$\Bbbk[\![x]\!]\langle y\rangle = \{\widehat{f}(x,y) \in \Bbbk[\![x,y]\!] \mid \exists s \in \mathbb{N}, \exists g(y,z) \in \Bbbk\langle y,z_1,\ldots,z_s\rangle,$$
$$\exists \widehat{z}_k(x) \in (x)\Bbbk[\![x]\!] \textit{ for } k=1,\ldots,s \textit{ such that } \widehat{f}(x,y) = g(y,\widehat{z}_1(x),\ldots,\widehat{z}_s(x))\}.$$

$\square$

References

[AvdP70] S. S. Abhyankar, M. van der Put, Homomorphisms of analytic local rings, *J. Reine Angew. Math.*, **242**, (1970), 26-60.

[An75] M. André, Artin theorem on the solution of analytic equations in positive characteristic, *Manuscripta Math.*, **15**, (1975), 314-348.

[Ar68] M. Artin, On the solutions of analytic equations, *Invent. Math.*, **5**, (1968), 277-291.

[Ar69] M. Artin, Algebraic approximation of structures over complete local rings, *Publ. Math. IHES*, **36**, (1969), 23-58.

[Ar71] M. Artin, Algebraic spaces, Yale Mathematical Monographs, 3. Yale University Press, New Haven, Conn.-London, 1971.

[Be77] J. Becker, Exposé on a conjecture of Tougeron, *Annales de l'Institut Fourier*, **27**, no. 4, (1977), 9-27.

[BM87] E. Bierstone, P. Milman, Relations among analytic functions II, *Ann. Inst. Fourier*, **37**, (1987), no.2, 49-77.

[EH77] P. M. Eakin, G. A. Harris, When $\Phi(f)$ convergent implies f convergent, *Math. Ann.*, **229**, (1977), 201-210.

[FB12] J. Fernández de Bobadilla, Nash Problem for surface singularities is a topological problem, *Adv. Math.*, **230**, (2012), 131-176.

[Ga71] A. M. Gabrielov, The formal relations between analytic functions, *Funkcional. Anal. i Prilovzen*, **5**, (1971), 64-65.

[Ga73] A. M. Gabrielov, Formal relations among analytic functions, *Izv. Akad. Naut. SSSR*, **37**, (1973), 1056-1088.

[Gr60] A. Grothendieck, Techniques de construction en géométrie analytique VI, *Séminaire Henri Cartan*, **13**, no. 1, (1960-61).

[Iz89] S. Izumi, The rank condition and convergence of formal functions, *Duke Math. J.*, **59**, (1989), 241-264.

[Mil78] P. Milman, Analytic and polynomial homomorphisms of analytic rings, *Math. Ann.*, **232**, (1978), no. 3, 247-253.

[Mir12] N. Mir, Algebraic approximation in CR geometry, *J. Math. Pures Appl. (9)*, **98**, No. 1, (2012), 72-88.

[Os16] W. F. Osgood, On functions of several complex variables, *Trans. Amer. Math. Soc.*, **17**, (1916), 1-8.

[Po86] D. Popescu, General Néron desingularization and approximation, *Nagoya Math. J.*, **104**, (1986), 85-115.

[Ro08] G. Rond, Approximation de Artin cylindrique et morphismes d'algèbres analytiques, *Proceedings of the Lêfest - Singularities I: Algebraic and Analytic Aspects*, Contemporary Mathematics, **474**, (2008), 299-307.

[Ro09] G. Rond, Homomorphisms of local algebras in positive characteristic, *J. Algebra*, **322**, no. 12, (2009), 4382-4407.

[Ro15] G. Rond, Artin Approximation, arXiv:1506.04717.

[SZ58] O. Zariski, P. Samuel, Commutative Algebra II, D. Van Nostrand Company, Inc., Princeton, New Jersey, (1960).

[Sh10] M. Shiota, Analytic and Nash equivalence relations of Nash maps, *Bull. Lond. Math. Soc.*, **42**, No. 6, (2010), 1055-1064.

[To76] J.-Cl. Tougeron, Courbes analytiques sur un germe d'espace analytique et applications, *Annales de l'Institut Fourier*, **26**, no. 2, (1976), 117-131.

[To90] J.-Cl. Tougeron, Sur les racines d'un polynôme à coefficients séries formelles, *Real analytic and algebraic geometry (Trento 1988)*, 325-363, *Lectures Notes in Math.*, **1420**, (1990).

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